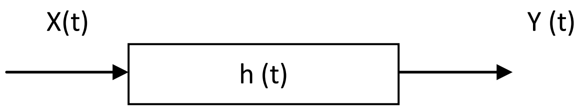
UNIT-5: LINEAR SYSTEMS RESPONSE TO RANDOM INPUTS

Consider a continuous LTI system with impulse response h (t). Assume that the system is always causal and stable. When a continuous time Random process X (t) is applied on this system, the output response is also a continuous time random process Y (t). If the random processes X and Y are discrete time signals, then the linear system is called a discrete time system. In this unit we concentrate on the statistical and spectral characteristics of the output random process Y (t).

System Response: Let a random process X (t) be applied to a continuous linear time invariant system whose impulse response is h(t) as shown in below figure. Then the output response Y (t) is also a random process. It can be expressed by the convolution integral, Y (t) = h (t) *X (t)



That is, the output response is Y (t) = $\int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau$.

Mean Value of Output Response: Consider that the random process X (t) is wide sense stationary process.

Mean value of output response=E[Y (t)], Then

E[Y (t)] = E [h (t) * X (t)]

=E [$\int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau$]

=$\int_{-\infty}^{\infty} h(\tau)E[X(t-\tau)]d\tau$

But $E[X(t-\tau)]$ = $\bar{X}$ =constant, since X (t) is WSS.

Then E[Y (t)] = $\bar{Y}$ = $\bar{X}\int_{-\infty}^{\infty} h(\tau)\, d\tau$. Also if H (ω) is the Fourier transform of h (t), then

H (ω) = $\int_{-\infty}^{\infty} h(\tau)e^{-j\omega t}\, dt$. At ω = 0, H (0) = $\int_{-\infty}^{\infty} h(t)\, dt$ is called the zero frequency response of the system. Substituting this we get E[Y (t)] = $\bar{Y}$ = $\bar{X}$ H (0) is constant. Thus the mean value of the output response Y (t) of a WSS random process is equal to the product of the mean value of the input process and the zero frequency response of the system.

Mean Square Value of Output Response:

Mean square value of output response is

$E[Y^2(t)] = E[(h(t) * X(t))^2]$

$= E[(h(t) * X(t))\,(h(t) * X(t))]$

$= E[\int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)X(t-\tau_2)d\tau_2]$

$= E[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t-\tau_1)X(t-\tau_2)h(\tau_1)h(\tau_2)d\tau_1 d\tau_2]$

$E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-\tau_1)X(t-\tau_2)]h(\tau_1)h(\tau_2)d\tau_1 d\tau_2$

Where τ_1 and τ_2 are shifts in time intervals. If input X(t) is a WSS random process then

$E[X(t-\tau_1)X(t-\tau_2)] = R_{XX}(\tau_1-\tau_2)$

Therefore $E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau_1-\tau_2)\,h(\tau_1)h(\tau_2)d\tau_1 d\tau_2$

This expression is independent of time t. And it represents the Output power.

Autocorrelation Function of Output Response: The autocorrelation of Y (t) is

$R_{YY}(\tau_1,\tau_2) = E[Y(t_1)\,Y(t_2)$

$= E[(h(t_1) * X(t_1))\,(h(t_2) * X(t_2))]$

$= E[\int_{-\infty}^{\infty} h(\tau_1)X(t_1-\tau_1)d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)X(t_2-\tau_2)d\tau_2]$

$= E[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t_1-\tau_1)X(t_2-\tau_2)h(\tau_1)h(\tau_2)d\tau_1 d\tau_2]$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t_1-\tau_1)X(t_2-\tau_2)]h(\tau_1)h(\tau_2)d\tau_1 d\tau_2$

We know that $E[X(t_1-\tau_1)X(t_2-\tau_2)] = R_{XX}(t_2-t_1+\tau_1-\tau_2)$.

If input X (t) is a WSS random process, Let the time difference $\tau = t_1-t_2$ and $t=t_1$ Then

$E[X(t-\tau_1)X(t+\tau-\tau_2)] = R_{XX}(\tau+\tau_1-\tau_2)$. Then

$R_{YY}(t,t+\tau) = R_{YY}(t,\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2)\,h(\tau_1)h(\tau_2)d\tau_1 d\tau_2$

If = $R_{XX}(\tau)$ is the autocorrelation function of X (t), then $R_{YY}(\tau) = R_{XX}(\tau) * h(\tau)\,h(-\tau)$

It is observed that the output autocorrelation function is a function of only τ. Hence the output random process Y(t) is also WSS random process.

Cross Correlation Function of Response:

If the input X (t) is WSS random process, then the cross correlation function of input X (t) and output Y(t) is

$R_{XY}(t, t+\tau)$ = E [X (t) Y (t + τ)]

$R_{XY}(\tau)$ = E [X (t)$\int_{-\infty}^{\infty} h(\tau_1)$ X (t + τ - τ_1) dτ_1]

$R_{XY}(\tau)$ =$\int_{-\infty}^{\infty}$ E [X (t) X (t + τ - τ_1)] $h(\tau_1)$dτ_1]

$R_{XY}(\tau)$ =$\int_{-\infty}^{\infty} R_{XX}(\tau - \tau_1)]\, h(\tau_1)d\tau_1$ which is the convolution of R_{XX}(τ) and h (τ).

Therefore $R_{XY}(\tau)$ = R_{XX}(τ) * h (τ) similarly we can show that $R_{YX}(\tau)$ = R_{XX}(τ) * h (-τ)

This shows that X (t) and Y(t) are jointly WSS. And we can also relate the autocorrelation functions and the cross correlation functions as

$R_{YY}(\tau)$ = R_{XY}(τ) * h (-τ)

$R_{YY}(\tau)$ = R_{YX}(τ) * h (τ)

Spectral Characteristics of a System Response: Consider that the random process X (t) is a WSS random process with the autocorrelation function R_{XX}(τ) applied through an LTI system. It is noted that the output response Y (t) is also a WSS and the processes X (t) and Y (t) are jointly WSS. We can obtain power spectral characteristics of the output process Y(t) by taking the Fourier transform of the correlation functions.

Power Density Spectrum of Response: Consider that a random process X (t) is applied on an LTI system having a transfer function H(ω). The output response is Y (t). If the power spectrum of the input process is S_{XX} (ω), then the power spectrum of the output response is given by S_{YY} (ω) = $|H(\omega)|^2$ S_{XX} (ω).

Proof: Let $R_{YY}(\tau)$ be the autocorrelation of the output response Y (t). Then the power spectrum of the response is the Fourier transform of $R_{YY}(\tau)$.

Therefore S_{YY} (ω) = F [S_{YY} (ω)]

= $\int_{-\infty}^{\infty} R_{YY}(\tau) e^{-j\omega\tau}\, d\tau$

We know that $R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2)\, h(\tau_1)h(\tau_2)d\,\tau_1 d\tau_2$

Then S_{YY} (ω) = $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2)\, h(\tau_1)h(\tau_2)d\,\tau_1 d\tau_2\, e^{-j\omega\tau}d\tau$

= $\int_{-\infty}^{\infty} h(\tau_1) \int_{-\infty}^{\infty} h(\tau_2) \int_{-\infty}^{\infty} R_{XX}(\tau + \tau_1 - \tau_2)\, e^{-j\omega\tau} d\tau\, d\tau_2\, d\tau_1$

= $\int_{-\infty}^{\infty} h(\tau_1) e^{j\omega\tau_1} \int_{-\infty}^{\infty} h(\tau_2) e^{j\omega\tau_2} \int_{-\infty}^{\infty} R_{XX}(\tau + \tau_1 - \tau_2)\, e^{-j\omega\tau} e^{j\omega\tau_1} e^{j\omega\tau_2} d\tau\, d\tau_2\, d\tau_1$

Let $\tau + \tau_1 - \tau_2 = t$, $d\tau = dt$

Therefore $S_{YY}(\omega) = \int_{-\infty}^{\infty} h(\tau_1) e^{j\omega\tau_1} d\tau_1 \int_{-\infty}^{\infty} h(\tau_2) e^{j\omega\tau_2} d\tau_2 \int_{-\infty}^{\infty} R_{XX}(t)\, e^{-j\omega t} dt$

We know that $H(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega t}\, dt$.

Therefore $S_{YY}(\omega) = H^*(\omega) H(\omega) S_{XX}(\omega) = H(-\omega)H(\omega) S_{XX}(\omega)$

Therefore $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$. Hence proved.

Similarly, we can prove that the cross power spectral density function is

$S_{XY}(\omega) = S_{XX}(\omega) H(\omega)$ and $S_{YX}(\omega) = S_{XX}(\omega) H(-\omega)$

Spectrum Bandwidth: The spectral density is mostly concentrated at a certain frequency value. It decreases at other frequencies. The bandwidth of the spectrum is the range of frequencies having significant values. It is defined as "the measure of spread of spectral density" and is also called rms bandwidth or normalized bandwidth. It is given by

$$W_{rms}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 S_{XX(\omega)} d\omega}{\int_{-\infty}^{\infty} S_{XX(\omega)} d\omega}$$

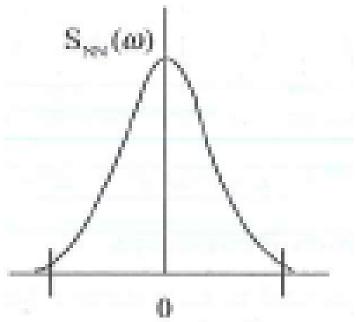
Types of Random Processes: In practical situations, random process can be categorized into different types depending on their frequency components. For example information bearing signals such as audio, video and modulated waveforms etc., carry the information within a specified frequency band.

The Important types of Random processes are;

1. Low pass random processes
2. Band pass random processes
3. Band limited random processes
4. Narrow band random processes

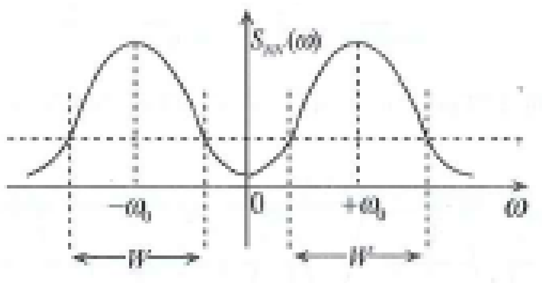
(1).Low pass random processes:

A random process is defined as a low pass random process X (t) if its power spectral density $S_{XX}(\omega)$ has significant components within the frequency band as shown in below figure. For example baseband signals such as speech, image and video are low pass random processes.

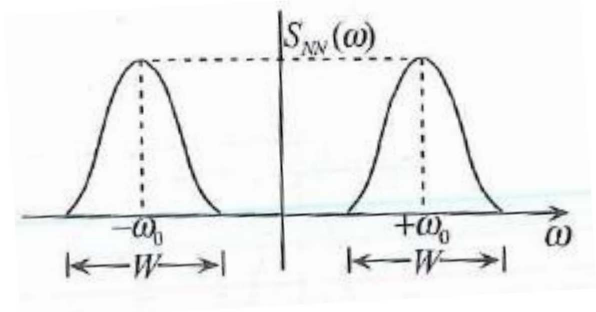


(2).Band pass random processes: A random process X (t) is called a band pass process if its power spectral density $S_{XX}(\omega)$ has significant components within a band width W that does not include ω =0. But in practice, the spectrum may have a small amount of power spectrum at ω =0, as shown in the below figure. The spectral components outside the band W are very small and can be neglected.

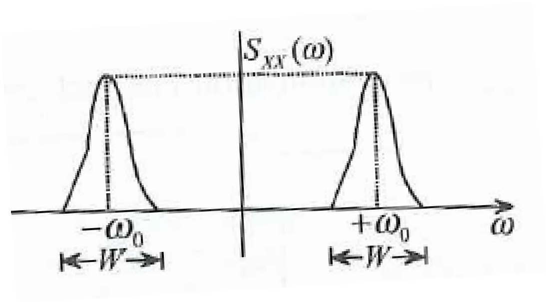
For example, modulated signals with carrier frequency ω_0 and band width W are band pass random processes. The noise transmitting over a communication channel can be modelled as a band pass process.



(3).Band Limited random processes: A random process is said to be band limited if its power spectrum components are zero outside the frequency band of width W that does not include ω =0. The power density spectrum of the band limited band pass process is shown in below figure.



(4).Narrow band random processes: A band limited random process is said to be a narrow band process if the band width W is very small compared to the band centre frequency, i.e. W<< ω_0, where W=band width and ω_0 is the frequency at which the power spectrum is maximum. The power density spectrum of a narrow band process N(t) is shown in below figure.



Representation of a narrow band process: For any arbitrary WSS random processes N(t), The quadrature form of narrow band process can be represented as N(t) = X(t) Cos ω_0t – Y(t)Sin ω_0t

Where X(t) and Y(t) are respectively called the in-phase and quadrature phase components of N(t). They can be expressed as

X (t) = A (t) Cos[Θ(t)]

Y (t) = A (t) Sin[Θ(t)] and the relationship between the processes A(t) and Θ(t) are given by

A (t) = $\sqrt{X^{2(t)} + Y^{2(t)}}$ and Θ (t) = $\tan^{-1}(\frac{Y(t)}{X(t)})$

Properties of Band Limited Random Processes: Let N (t) be any band limited WSS random process with zero mean value and a power spectral density, $S_{NN}(\omega)$. If the random process is represented by

N (t) = X (t) Cos ω_0t – Y(t)Sin ω_0t then some important properties of X (t) and Y (t) are given below

1. If N (t) is WSS, then X (t) and Y (t) are jointly WSS.
2. If N (t) has zero mean i.e. E [N(t)] = 0, then E [x (t)] = E [Y (t)] = 0
3. The mean square values of the processes are equal i.e. E [N^2(t)] = E [X^2(t)] = E [Y^2(t)].
4. Both processes X (t) and Y (t) have the same autocorrelation functions i.e. $R_{XX}(\tau) = R_{YY}(\tau)$.
5. The cross correlation functions of X (t) and Y (t) are given by $R_{YX}(\tau) = -R_{YX}(\tau)$. If the processes are orthogonal, then $R_{YX}(\tau) = R_{YX}(\tau)$ = 0.
6. Both X (t) and Y (t) have the same power spectral densities

   $S_{YY}(\omega) = S_{XX}(\omega) = \begin{cases} S_{N(\omega-\omega_0)+S_{N(\omega+\omega_0)}} & for\, |W| \le \omega_0 \\ 0 & \end{cases}$
7. The cross power spectrums are $S_{XY}(\omega)$ = -$S_{YX}(\omega)$.

8. If N (t) is a Gaussian random process, then X (t) and Y (t) are jointly Gaussian.
9. The relationship between autocorrelation and power spectrum S_{NN} (ω) is

   $R_{XX}(\tau)$ = $\frac{1}{\pi}\int_0^{\infty} S_{NN(\omega)\cos[(\omega-\omega_0}) \tau] d\omega$ and

   $R_{YY}(\tau)$ = $\frac{1}{\pi}\int_0^{\infty} S_{NN(\omega)\cos[(\omega-\omega_0}) \tau] d\omega$

10. If N (t) is zero mean Gaussian and its psd, S_N(ω) is symmetric about $\pm\omega_0$, then X (t) and Y (t) are statistically independent.